# Precalculus Refresher 

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## 0 Guidelines

This Precalculus Refresher course will serve as an essential guide as you review and refresh your precalculus skills. This is not a comprehensive precalculus resource and cannot replace the contents of a precalculus course. If you do not feel comfortable with most of the content presented here and find it hard to follow these materials, we strongly advise you to talk to your instructor about your calculus readiness and find out what options are available to you if you are not as prepared as needed.

The main purpose of this self-paced course is to serve as a concise targeted refresher guide for students who have taken precalculus. There are 7 Modules, each focusing on specific topics necessary in calculus. Topics and their coverage range from basic to advanced, depending on the fluency in those topics that instructors have historically observes in calculus students. While we recommend you review all the topics in full, you will likely need to spend more time on some topics than others.
Each Module consists of 3 main parts: skeleton notes, corresponding videos, exercises in the notes. The original course offering in fall 2020 was also supported by online homework exercises on WebWork and monitored Piazza discussion forum; these are currently not available, so please ignore their mention (if any) in the videos. Here are the main videos:

- Module 1, part a: fractional arithmetic
- Module 1, part b: exponents and exponential laws, rationalizing
- Module 1, part c: simplifying expressions by factoring
- Module 1, part d: solving various types of equations
- Module 2: graphing fundamentals
- Module 3: functions
- Module 4, part a: linear functions
- Module 4, part b: quadratic, power and polynomial functions
- Module 4, part c: rational functions
- Module 5, part a: exponential and logarithmic functions
- Module 5, part b: applications of exponential and logarithmic functions
- Module 6, part a: introduction to trig functions
- Module 6, part b: triangle trigonometry
- Module 6, part c: inverse trigonometric functions
- Module 6, part d: solving trigonometric equations
- Module 7: applications

If you have any access issues or any questions, please email Kseniya.Garaschuk@ufv.ca. You must email form your student account and include "Precalculus Refresher" in the subject line.

## 1 Algebraic fundamentals

- fractional arithmetic
- integer and rational exponents
- simplifying expressions

Video for Module 1, part a: fractional arithmetic
$\underline{\text { Multiplying fractions }}$
$\frac{2}{3} \cdot \frac{5}{7}=$
$\frac{2}{3} \cdot\left(-\frac{5}{7}\right)=$

Cancelling common factors
$\frac{24}{30}=$
$\frac{10 x+5}{5}=$
$\frac{4}{3} \cdot \frac{5}{8}=$
$\frac{h}{h^{2}+h}=$
$\frac{2}{18}=$
$\frac{5 x+5}{x+5}=$

Dividing fractions

$$
\begin{aligned}
& \frac{2}{3} \div \frac{5}{7}= \\
& \frac{2}{3} \div\left(-\frac{5}{7}\right)=
\end{aligned}
$$

$$
\frac{2}{3} \div 5=
$$

$$
\frac{x y^{2}}{z} \div \frac{x^{2} y}{z^{2}}=
$$

$$
\frac{2}{7}+\frac{5}{7}=
$$

$$
\frac{2}{3}-\frac{7}{5}=
$$

$$
\frac{2}{3}+5=
$$

$$
\begin{aligned}
& \frac{1}{8}-\frac{1}{4}= \\
& \frac{x y}{z}-\frac{z}{5}= \\
& \frac{2}{x}+\frac{3}{y}=
\end{aligned}
$$

$\underline{\text { Combining operations }}$

$$
\frac{3}{4} \cdot \frac{2}{3}-\frac{2}{6}=
$$

$$
\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=
$$

$$
\frac{3}{4} \cdot\left(\frac{2}{3}-\frac{2}{6}\right)=
$$

$$
\frac{a+b}{a}+\frac{1}{a^{2}}=
$$

$$
\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}=
$$

Video for Module 1, part b: exponents and exponential laws, rationalizing
Exponents

| $2^{3}$ |  |  |
| :--- | :--- | :--- |
| $2^{2}$ |  |  |
| $2^{1}$ |  |  |
| $2^{0}$ |  |  |
| $2^{-1}$ |  |  |
| $2^{-2}$ |  |  |
| $2^{-3}$ |  |  |


| $3^{2} \cdot 3^{5}$ |  |  |
| :---: | :--- | :--- |
| $2^{3} \cdot 8^{2}$ |  |  |
| $\frac{5^{7}}{5^{3}}$ |  |  |
| $\left(7^{2}\right)^{3}$ |  |  |
| $\frac{4^{3}}{8^{2}}$ |  |  |
| $(x \cdot y)^{2}$ |  |  |
| $(x+y)^{2}$ |  |  |

Fractional powers, roots/radicals

$$
\begin{array}{ll}
\frac{4^{-1}\left(x^{2} y^{3}\right)^{2}}{x^{-2} y^{5}}= & 2 a^{-1}+(4 a)^{-1}= \\
\left(\frac{x^{-3}}{x^{5}}\right)^{-2}= & (x+y)^{-1}= \\
\left(x^{2}+y^{5}\right)^{3}= & \left(x^{-1}+y^{-1}\right)^{-1}= \\
\sqrt{144}= & 8^{2 / 3}= \\
\sqrt{-64}= & \sqrt{25 x^{4}}= \\
\sqrt[3]{-64}= & \sqrt{4 x^{2}+9 x^{4}}= \\
\sqrt{\frac{1}{9}}= & \left(81 x^{2}-4 y^{2}\right)^{-1 / 2}= \\
& \\
\sqrt{x^{2}+4}+\frac{1}{\sqrt{x^{2}+4}}= & \\
\frac{\sqrt{3 x+1}-3 x(3 x+1)^{-\frac{1}{2}}}{(\sqrt{3 x+1})^{2}}= &
\end{array}
$$

Rationalizing

$$
\begin{aligned}
& \frac{1}{2 \sqrt{6}}= \\
& \frac{3}{5+\sqrt{3}}=
\end{aligned}
$$

$$
\frac{1}{2 \sqrt[3]{6}}=
$$

## Exercises.

1. Evaluate the following:
(a) $\sqrt{0}$
(c) $\sqrt[3]{-64}$
(e) $8^{5 / 3}$
(b) $\sqrt{-64}$
(d) $(9 / 25)^{-1 / 2}$
(f) $\sqrt{200} / \sqrt{100}$
2. Write each of the following as a single radical:
(a) $\sqrt{3} \cdot \sqrt{6}$
(c) $\sqrt{2}+\sqrt{8}$
(b) $2 \sqrt{5}$
(d) $2 \sqrt[4]{3} \cdot \sqrt[2]{5}$
3. Rationalize the denominator and simplify as far as possible in each of the following:
(a) $\frac{1}{\sqrt{x}-\sqrt{x-1}}$
(b) $\frac{x+13}{x+\sqrt{13}}$
(c) $\frac{x-27}{\sqrt{x+9}-6}$
4. Without using a calculator, simplify $\sqrt[3]{9 \cdot 9 \cdot 9}$ and $\sqrt[3]{9} \cdot \sqrt[3]{9} \cdot \sqrt[3]{9}$.
5. Explain why $\sqrt[3]{x^{6}}=(\sqrt[3]{x})^{6}$.
6. Without using your calculator, decide which of the following is larger:
(a) $\sqrt{13}$ or $\sqrt[3]{26}$
(b) $\sqrt{15}$ or $\sqrt[4]{68}$
(c) $\sqrt[3]{26}$ or $\sqrt[4]{80}$
7. Suppose the number of mini chocolate bars Katerina eats $t$ days after Halloween can be modelled by the function $C(t)=27 \cdot 3^{-t}$. How many candy bars did she eat on Halloween? Is Katerina eating more or less bars each day after Halloween? Justify.
8. Peter, Kseniya, Paul, Jane, and Micheal are competing to build the tallest Lego tower, but they need your help to determine who won! Unfortunately, each of them used a ruler with a strange scale to measure the height of their tower. The heights they reported were $\sqrt{15} \mathrm{ft}, \sqrt[3]{25} \mathrm{ft}, \sqrt[4]{100} \mathrm{ft}, \sqrt[5]{31 \mathrm{ft}}, \sqrt[6]{81} \mathrm{ft}$, respectively. Your mission, if you choose to accept it, is to order the tower heights from smallest to largest. You need to have your reasoning ready, this group is very competitive, will want explanations and do not trust calculators.
9. Simplify:
(a) $\left(x^{2}+1\right)^{-\frac{1}{2}}(x+2)+3\left(x^{2}+1\right)^{\frac{1}{2}}$
(b) $5(2 x-1)^{\frac{1}{3}}-10 x(2 x-1)^{-\frac{2}{3}}$

## Video for Module 1, part c: simplifying expressions by factoring

Simplifying expressions by factoring
Find the largest common factor in each of the following expressions:

$$
a^{2} b^{2} c \text { and } b^{3} c^{5} \quad \frac{3 x^{2} z}{y} \text { and } \frac{15 x z^{3}}{4 y^{2}} \quad 15 a^{3} b c, 3 a b c, \text { and } 5 b^{2} c^{2}
$$

Factor each of the following expressions:

$$
\begin{array}{ll}
a^{2} b^{2} c+b^{3} c^{5}= & x^{2 / 5}-3 x^{1 / 5}= \\
9 x^{4}-3 x^{2}= & x^{2}(2-4 x)+2 x(4-8 x)= \\
5(x-2)^{2}+6(x-2)= &
\end{array}
$$

## Useful formulas

Factor each of the following expressions:
$81 s^{2}-t^{4}=$
$x^{2}+y^{2}=$

$$
{ }^{2}+9
$$

$$
x^{2}-x-6=
$$

.

$$
x^{4}-x^{2}-6=
$$

Simplify as far as possible by factoring and cancelling:
$\frac{x^{2}+x-6}{x-2}=$

$$
\frac{x^{2}-3}{x+\sqrt{3}}=
$$

$$
\frac{x^{3}+9 x^{2}}{-2 x-18}=
$$

$$
\frac{x+2}{x^{2}+4}=
$$

$$
\frac{x^{3}-6 x^{2}+9 x}{x-3}=
$$

$$
\frac{3 x+3}{x+3}=
$$

$$
\frac{25-x^{2}}{x-5}=
$$

## Exercises.

1. Simplify the following expressions as far as possible:
(a) $\frac{3}{4} \cdot \frac{2}{3}-\frac{2}{6}$
(e) $-3^{2}$
(i) $\frac{x}{x+1}-\frac{x}{x^{2}+2}$
(b) $\frac{3}{4} \cdot\left(\frac{2}{3}-\frac{2}{6}\right)$
(f) $(-3)^{2}$
(c) $\frac{1}{y-1}-\frac{1}{y+1}$
(g) $\frac{x^{-1}+y^{-1}}{(x y)^{-1}}$
(j) $\frac{x}{x+1}-\frac{x}{x^{2}+2 x}$
(d) $\sqrt{x^{2}+y^{2}}$
(h) $\frac{1}{x+2}+\frac{2}{x^{2}-4}$
(k) $\frac{\frac{x}{x+2}+3}{\frac{x+1}{x-1}}$
2. Show that the following expressions are NOT true; that is, they might hold for some very specific values but not in general. First, to persuade yourself, find some values where these expressions are false. Next, for each left-hand side, find the correct expression for the right-hand side.
(a) $\frac{1}{a}+\frac{1}{b}=\frac{1}{a+b}$
(c) $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$
(d) $(a+b)^{2}=a^{2}+b^{2}$
(b) $\sqrt{a^{2}-b^{2}}=a-b$
(e) $x^{a+b}=x^{a}+x^{b}$
3. Find a number $a$ such that the expression $\frac{3 x^{2}+a x+a+3}{x^{2}+x-2}$ can be simplified by factoring.

Video for Module 1, part d: solving various types of equations
Solving equations

## Linear equations

$3 x+9=17$

$$
2(x+4)=3(x+1)
$$

## $\underline{\text { Quadratic equations }}$

$$
x^{2}-3 x+2=0
$$

$$
2 x^{2}-8 x+3=0
$$

$x^{2}-2 x+5=0$

$$
x^{2}-9=0
$$

$$
x+\sqrt{x}-6=0
$$

# Other equations 

$$
\frac{4}{\sqrt{x^{3}}}=\frac{1}{\sqrt{x}}
$$

$$
\sqrt{2 x-1}=\sqrt{x-1}+1
$$

$$
(x-2)(x+3)^{2}\left(x^{2}-16\right)=0
$$

$$
x^{3}=4 x
$$

$$
\sqrt{x}-3=5+2 \sqrt{x}
$$

$$
\frac{1}{x-5}+\frac{1}{x+5}=\frac{10}{x^{2}-25}
$$

## Exercises.

1. Solve each of the following for $x$ :
(a) $x^{4}-5 x^{2}+4=0$
(c) $x^{6}+6 x^{3}-16=0$
(e) $x^{2 / 5}-3 x^{1 / 5}+2=0$
(g) $\frac{x^{2}-4}{\sqrt{x-3}}=0$
(b) $x^{4}-5 x^{2}-36=0$
(d) $x+\sqrt{x}=6$
(f) $\frac{x^{2}-9}{x+2}=0$
(h) $k x^{2}-x=0$
2. Which of the following equations have identical sets of solutions? Explain why.
(a) $x=2$
(c) $\frac{x-2}{x-2}=0$
(e) $x(x-4)=4$
(b) $4-x^{2}=0$
(d) $x^{2}-4=0$
(f) $x^{3}-8=0$
3. Amanda asks her class the following question: "Find all solutions to the equation $x^{2}=\sqrt{5} x$." Chris divides both sides by $x$ and is left with $x=\sqrt{5}$, which he claims is the only solution. Explain why Chris is wrong.

## 2 Graphing fundamentals

- main graph features: positive, negative, increasing, decreasing
- sketching from table of values
- intercepts and their algebraic meaning
- piecewise functions

Video for Module 2: graphing fundamentals
Amanda fills in a bathtub, gets in, relaxes for 30 minutes, gets out and drains the water. Sketch a graph showing how the height of the water varies over time. Sketch a graph showing how the volume of the water varies over time.

The graph of a function $f$ is given below. List the $x$ and $y$-intercepts, if any. Determine the intervals where the function is positive and negative, increasing and decreasing.


The graph of a function $f$ is the set of points which satisfy the equation $y=f(x)$.
Graph $f(x)=x^{2}-x-6$.

Sketch the graphs of $x^{2}, x^{3}, x^{4}, x^{5}$. How do these graphs behave near the origin and for large values of $x$ ?

Sketch the graphs of $x^{2}, 3 x^{2}, \frac{1}{2} x^{2},-2 x^{2}$. Sketch $x^{2}+3$ and $x^{2}-4$. Sketch $(x+3)^{2}$ and $(x-4)^{2}$.

Sketch $y=3 x^{4}$ and $y=5 x^{7}$. Find the points of intersection.
$\underline{\text { Piecewise functions }}$
Sketch the graph of $f(x)= \begin{cases}x^{3}, & x \leq 1 \\ x^{2}, & x>1 .\end{cases}$
$\underline{\text { Reading information from the graph }}$
$u(x)$ :


$w(x)$ :


Evaluate:

$$
u(-1)=\quad u(4)=\quad v(0)=\quad v(2)=\quad w(1)=\quad w(2)=
$$

Find the values of $x$ where:

$$
u(x)=2 \quad u(x) \geq 2 \quad v(x)=0 \quad w(x)=2
$$

Find the zeros (if any) of $u(x)$ :
Find the zeros (if any) of $v(x)$ :
Domain and range

Find the domain and range of $y=x^{2}$.

Find the domain and range of this function:


Consider the following function $y=f(x)$ given by its graph on the right:
Find the domain of $f(x)$.
Find the range of $f(x)$.
List the $x$-intercepts, if any exist.
List the $y$-intercepts, if any exist.
Find the zeros of $f(x)$.
Solve $f(x)<0$.
Solve $f(x) \geq 2$.
Solve $f(x) \geq x^{2}$.
Determine $f(2)$.
Solve $f(x)=-3$.
Find the number of solutions to $f(x)=1$.
List the intervals on which $f$ is increasing.

Algebraic meaning of intercepts.
Determine the $x$-intercepts and $y$-intercepts for each of the following functions:

$$
f(x)=x^{2}-2 x
$$

$$
g(x)=x^{2}+2
$$

$$
h(x)=(x+1)^{2}
$$

## Exercises.

1. (a) Sketch the graph of a function that always increases and is never negative.
(b) Sketch the graph of a function that always increases and is never positive.
2. Sketch the graphs that describe how the following situations change over time:
(a) The average daily temperature in Abby over a two year period.
(b) The height of a human being over a lifetime from birth to death.
(c) The average number of cars per hour passing a given point on a highway during a 24 hour period.
3. Consider the following function $y=f(x)$ :

(a) Find the domain of $f(x)$.
(b) Find the range of $f(x)$.
(c) List the $x$-intercepts, if any exist.
(d) List the $y$-intercepts, if any exist.
(e) Find the zeros of $f(x)$.
(f) Solve $f(x) \geq 0$.
(g) Solve $f(x) \leq-1$.
(h) Solve $f(x) \leq x^{2}$.
(i) Determine $f(-2)$.
(j) Solve $f(x)=-3$.
(k) Find the number of solutions to $f(x)=1$.
(l) List the intervals on which $f$ is increasing.
4. Sketch $y=2 x^{2}$ and $y=6 x^{6}$. Find the points of intersection.
5. Sketch $y=x^{3}, y=x^{1 / 3}, y=x^{-3}$.
(a) Which of these functions increases most steeply for values of $x$ greater than 1 ?
(b) Which decreases for large values of $x$ ?
(c) Which of these functions are not defined at $x=0$ ?
(d) Which functions are not defined for negative $x$ values?
(e) Compare the values of these functions for $0<x<1$.
6. Repeat Exercise 5 for functions $y=x^{4}, y=x^{1 / 4}, y=x^{-4}$.

## 3 Functions

- various ways to represent functions
- function notation
- domain and range
- combining and composing functions
- inverse functions


## Video for Module 3: functions

A function is a rule that assigns to each element of a set exactly one element of another set.
$\underline{\text { Modeling with functions }}$
Every tablet of Tylenol contains 325 mg of acetaminophen. Approximately $67 \%$ of the drug is removed from the body every 4 hours. Kseniya has taken an initial dose of $x \mathrm{mg}$ of acetaminophen four hours ago and just swallowed two more regular tablets of Tylenol.
a) Write a function describing the amount of acetaminophen in Kseniya's system.
b) What is the implied domain of the function in part a)?
c) Write a function describing the amount of acetaminophen in Kseniya's system four hours from now.

## Function notation

Given $f(x)=x^{2}-2 x+3$ and $g(x)=\sqrt{x+3}$, evaluate:
$f(3)=$
$f(10)=$
$f(t)=$
$g(3)=$
$g(-10)=$
$f(t+2)=$

Given $h(x)=\left\{\begin{array}{l}x^{2}-3, \\ x^{3}+5, \\ x>0\end{array} \quad\right.$, evaluate

$$
h(3)=
$$

$h(-5)=$
$h(0)=$

Domain of a function
The domain of a function is the set of all $x$ values that can be plugged into the function equation that result in a real number. The range is the set of all $y$ values that can ever result as an output.

Determine the domains of the following functions:

$$
f(x)=\frac{x}{x+3} \quad g(x)=\sqrt{x-3} \quad h(x)=\frac{\sqrt{5 x+3}}{x^{2}+4}
$$

## Inverse functions

Verbally:

Symbolically:

Analytically:

Graphically:

## Operations on functions

Standard operations:

$$
\begin{aligned}
(f+g)(x) & =f(x)+g(x) \\
(f-g)(x) & =f(x)+g(x) \\
(f \cdot g)(x) & =f(x)+g(x) \\
(f \div g)(x) & =f(x) \div g(x), \quad g(x) \neq 0
\end{aligned}
$$

Function composition:

Let $f(x)=x^{3}-2 x+1$ and $g(x)=\sqrt{x+2}$. Evaluate

$$
\begin{array}{ll}
(f+g)(1)= & f(g(x))= \\
(2 f-3 g)(4)= & g(f(x))= \\
(f / g)(2)= & f(s+7)= \\
& \\
(f(g(-3))= &
\end{array}
$$

Consider the following table of function values for $h(x)$ and $h(x)$.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $h(x)$ | -1 | 7 | 2 | 0 | -3 | 8 |
| $k(x)$ | 8 | -12 | 9 | 5 | 34 | 1 |

Evaluate

$$
\begin{array}{ll}
(h \circ k)(5)= & h^{-1}(g(0))= \\
(k \circ h)(2)= & k\left(h^{-1}(0)\right)= \\
h^{-1}(-3)= & k^{-1}\left(h^{-1}(8)\right)=
\end{array}
$$

## Graphing functions

A function establishes a relationship between an input and an output.
Graph is a representation of that relationship.

Sketch the graphs of the following common functions: $x, x^{2}, x^{3}, x^{4}, x^{5}, \sqrt{x}, \sqrt[3]{x}, \frac{1}{x}, \frac{1}{x^{2}},|x|, 2^{x}$ on the next page. Consider the graph of $w(x)$ :


Find the domain of $w(x)$ :
Find the range of $w(x)$ :

State the $x$-intercepts:

State the $y$-intercepts:

Evaluate:

$$
\begin{array}{ll}
w(1)= & w(w(2))= \\
w(-1)= & w(w(-3))= \\
w(0)+w(3)= & w^{-1}(4)= \\
\frac{w(0)}{w(3)}= & w^{-1}(1)=
\end{array}
$$

Vertical and horizontal line tests













## Exercises.

Use the following functions for questions 1-5:

$$
\begin{array}{ccc}
f(x)=\frac{9 x^{2}+2}{3 x} & g(x)=2 \sqrt{4 x-3} \quad h(x)=|2 x+3|+2 & k(x)=\frac{-2 x^{2}+x}{x-2} \\
p(x)=\frac{\sqrt{9 x+18}}{x-3} & q(x)= \begin{cases}\frac{1}{3} x, x \leq 0 \\
2-x^{2}, x>0\end{cases} & r(x)=-11 x^{3}+2 x
\end{array}
$$




1. Evaluate:
(a) $f(-3)$
(d) $k(-1)$
(g) $q(7)$
(j) $v(-1)$
(m) $g(f(2))$
(b) $g(7)$
(e) $p(0)$
(h) $r\left(\frac{1}{2}\right)$
(k) $(f+g)(3)$
(n) $h(-2 u(0))$
(c) $h(-8)$
(f) $q(0)$
(i) $u(3)$
(l) $f(g(2))$
(o) $u(u(4))$
2. Simplify as much as possible:
(a) $k(x+h)$
(b) $r(-x)$
(c) $g\left(x^{2}\right)$
(d) $g\left(x^{2}+1\right)$
(e) $g\left((x+1)^{2}\right)$
3. Find the zeros (if any) of all the functions: $f(x), g(x), h(x), k(x), p(x), q(x), r(x), u(x), v(x)$.
4. Find domains of all the functions: $f(x), g(x), h(x), k(x), p(x), q(x), r(x), u(x), v(x), f(g(x)), g(h(x))$.
5. Find (or explain why it does not exist):
(a) $u^{-1}(1)$
(b) $v^{-1}(4)$
(c) $\frac{1}{u(1)}$
(d) $\frac{1}{v(4)}$
(e) $f^{-1}(0)$
(f) $g^{-1}(0)$
6. Consider the following table of function values for $a(x)$ and $b(x)$.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $a(x)$ | -1 | 7 | 2 | 0 | -3 | 8 |
| $b(x)$ | 8 | -12 | 9 | 5 | 34 | 1 |

(a) $a(0)$
(c) $(a \cdot b)(5)$
(e) $\frac{b}{a}(3)$
(g) $b^{-1}(8)$
(i) $b\left(a^{-1}(0)\right)$
(b) $(a+b)(4)$
(d) $\frac{a}{b}(3)$
(f) $a^{-1}(-3)$
(h) $a^{-1}(b(0))$
7. Find the domain of the following functions:
(a) $a(x)=\frac{1}{x+2}$
(b) $b(x)=\frac{1}{\sqrt{x+2}}$
(c) $c(x)=\sqrt{-x}$
(d) $d(x)=\frac{x+2}{x^{2}-4}$
8. Determine the domains of the following functions: $g(x)=\sqrt{2 x-4}, k(x)=\sqrt[3]{2 x-4}, p(x)=\sqrt[n]{2 x-4}$.
9. Let $f(x)=x^{2}, g(x)=-x$, and $h(x)=\sqrt{x}$. Compute each of the following and determine their domains.
(a) $g(f(x))$
(b) $f(g(x))$
(c) $h(g(x-2))$
(d) $g(h(2 x))$
10. (a) The function $m(t)$ gives the number of monkeys on an island, where $t$ is measures in years starting from 2000. The function $c(m)$ gives the number of coconuts eaten by $m$ monkeys. What does the composition $c(m(t))$ represent?
(b) The function $m(t)$ gives the number of monkeys on an island, where $t$ is measures in years starting from 2000. here are some of its values:

$$
m(0)=15, m(1)=23, m(2)=33, m(3)=37, m(4)=41, m(5)=47, m(6)=65
$$

Explain why $m(t)$ has an inverse function. Find $m^{-1}(41)$ and explain what the answer means. What does $m^{-1}(t)$ represent in general?
11. Kseniya buys a dog house for Schumi and has to get it delivered to her house. Her total bill includes the price of the dog house, $7 \%$ tax and the delivery fee of $\$ 50$.
(a) Write a function $t(x)$ representing Kseniya's total bill after taxes on the purchase amount $x$.
(b) Write a function $f(x)$ for the total bill after taxes and delivery fee on the purchase amount $x$.
(c) Calculate and interpret $t(f(x))$ and $f(t(x))$. Which will result in lower cost for Kseniya?
(d) Suppose that taxes cannot be charges on delivery fees. Which composite function, $t(f(x))$ or $f(t(x))$, must be used?
12. David is looking to buy a new TV. He has a $\$ 50$-off coupon and the store currently has a $10 \%$ off sale on all new appliances.
(a) Write a function $c(x)$ representing the total cost of the TV after applying the coupon on a TV that costs $x$ dollars.
(b) Write a function $s(x)$ representing the total cost of the TV after applying the $10 \%$ off promotion to a TV that costs $x$ dollars.
(c) Calculate and interpret $c(s(x))$ and $s(c(x))$.
(d) Suppose David's favourite TV cost $\$ 1000$. Should he use the coupon first and then take the $10 \%$ off or should he take $10 \%$ off and then apply the coupon?
13. Consider the following function:

$$
k(x)=\left\{\begin{array}{l}
2 x-2, \quad x<1 \\
x^{2}-3, \quad x \geq 1
\end{array}\right.
$$

(a) Find (or explain why it does not exist): $k(0), k(1), k(2), k(4), k^{-1}(0), k^{-1}(1), k^{-1}(2), k^{-1}(4)$.
(b) Sketch the graph of $k(x)$.
(c) Explain why the inverse of $k(x)$ exists for all real numbers.
(d) Find the inverse of $k(x)$ algebraically.
(e) Sketch the graph of the inverse of $k(x)$.
14. Think of eight people you know who drive. Define a function which assigns to each person the brand of car they drive (e.g. Ford, Toyota, Ferrari, etc.). Is your function invertible? If yes, specify the inverse function. If no, explain why not. If you found an inverse function, change your original function so that it no longer has an inverse. If your function didn't have an inverse, what modification or restriction could you make so that it would have an inverse function?

## 4 Linear functions, polynomial functions and rational functions

- general forms
- domain and range
- graphs
- intercepts
- asymptotes

Video for Module 4, part a: linear functions
Lines
A linear equation in two variables is an equation that can be written as $a x+b y=c$, where $a, b$, and $c$ are real numbers and $a, b$ cannot both be zero.

A linear function is a function of the form $f(x)=m x+b$, where $m$ and $b$ are real numbers. The graph of a linear function is a line.

Equations of lines:
standard form slope-intercept form point-slope form

Slope

Vertical lines

Horizontal lines

Parallel lines

Perpendicular lines

Domain and range

Graphing lines

Sketch and find an equation of the line through the point $(1,1)$ with the slope of 3 .

Sketch and find an equation of the line through the points $(1,1)$ and $(2,5)$.

Sketch and find an equation of the line through the point $(3,1)$ and parallel to the line $2 x+3 y=7$.

A car leaves a town at 60 kilometers per hour. How long will it take a second car, travelling at 90 kilometers per hour, to catch the first car if it leaves 1 hour later?

## Exercises.

1. For each of the following, find an equation of the line and sketch it.
(a) A line passing through the point $(2,3)$ with the slope of -4 .
(b) A line passing through the points $(-1,1)$ and $(2,-5)$.
(c) A line passing through the point $(3,1)$ and perpendicular to the line $2 x+3 y=7$.
(d) A line passing through the point $(-2,4)$ with $x$-intercept of 3 .
(e) A line parallel to the $y$-axis passing through the point $(2,-3)$.
2. See table on the next page. Fill in all the blank cells.
3. Sketch and find an equation of the line going through $(-1,2)$ that is also
(a) parallel to the $x$-axis;
(b) parallel to the $y$-axis;
(c) perpendicular to the line $x+y=1$;
(d) has a slope of 17 .
4. (a) Abbotsford International airport parkade charges $\$ 2$ for the first hour, $\$ 1$ for each subsequent hour or a portion of it for a maximum of $\$ 9$ per day ( 24 hour period). Sketch the graph of the cost of parking as a function of time parked.
(b) Abbotsford Mobile cell phone provider charges $\$ 2$ for the first minute of a long-distance call and $\$ 1$ for each subsequent minute pro-rated (that is, they charge 50 cents for 30 additional seconds, 25 cents for 15 seconds and so on). Sketch the graph of the cost of a long-distance call as a function of the length of the call.
(c) Give formulas for the two functions above.
5. The temperature scales Celsius and Fahrenheit have a linear relationship with $0^{\circ} \mathrm{C}=32^{\circ} \mathrm{F}$ and $100^{\circ} \mathrm{C}=$ $212^{\circ} \mathrm{F}$.
(a) Write a linear equation expressing $F$ in terms of $C$.
(b) If it is $75^{\circ} \mathrm{C}$, what is the temperature in $F$ ? What about if it is $175^{\circ} \mathrm{F}$ ?
(c) If it is $75^{\circ} F$, what is the temperature in $C$ ? What about if it is $-25^{\circ} F$ ?
(d) Did you compute each value in parts (b) and (c) separately? Could you have avoided this?
(e) Sketch the functions $C(F)$ and $F(C)$.

Linear Functions: fill in the blanks to complete each row

| Graph(s) | Slope | Standard form: $a y+b x=c$ | SlopeIntercept form: $y=m x+b$ | $\begin{aligned} & x \text { and } y \\ & \text { intercepts } \end{aligned}$ | Additional Info |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $2 y-4 x=2$ |  |  |  |
|  |  |  |  |  |  |
|  | $m=0$ |  |  |  | Passes through $(4,2)$ |
|  |  |  |  |  | Parallel to $y=\frac{1}{2} x-3$ |
|  |  |  |  |  | Perpend. to $2 x+3 y=4$ |

Video for Module 4, part b: quadratic, power and polynomial functions
Quadratic functions
A quadratic function is a function of the form $f(x)=a x^{2}+b x+c$, where $a, b$ and $c$ are real numbers. The graph of a quadratic function is a parabola.

Sketch the graph of $f(x)=(x+1)^{2}-3$.

Sketch the graph of $f(x)=x^{2}+3 x-4$.

Domain and range

Power functions
Power functions are functions of the form $f(x)=k x^{n}$, where $n$ is a positive integer and $k$ is a non-zero constant. Sketch the graphs of $y=x^{2}, x^{3}, x^{4}, x^{5}, \ldots$. How would a coefficient affect the shape?


On the same graph, sketch $y=3 x^{4}$ and $y=5 x^{7}$. Find the points of intersection.

Polynomial functions are functions of the form $f(x)=a_{n} x^{n}+a_{n-1} x^{n-1}+\cdots a_{0}$, where $a$ 's are constants. End-behavior: what happens to $f(x)$ as $x$ approaches infinity? negative infinity?

Sketch $y=x^{3}-4 x$.

Sketch $y=(3 x-2)(x+3)(x-4)$.

Domain and range

## Exercises.

1. See two tables on the next two pages. Fill in all the blank cells.
2. Graph each of the following. Your graph should show the maximum and minimum values of $y$, the $x$ and $y$ intercepts, and the axis of symmetry. Determine the domain and range of each of the graphs.
(a) $y=x^{2}$
(c) $y=(x-1)^{2}$
(e) $y=-(x-1)^{2}$
(b) $y=x^{2}+1$
(d) $y=(x-1)^{2}-4$
(f) $y=(-x+1)^{2}$
3. Graphically determine the $x$-value(s) for which
(a) $(-x+1)^{2} \geq 0$
(b) $(-x+1)^{2} \leq 1$
(c) $1 \leq(-x+1)^{2} \leq 4$
4. The height above ground of a toy rocket launched upward from the top of a building is given by

$$
h=-16 t^{2}+96 t+256
$$

(a) What is the height of the building?
(b) What is the maximum height attained by the rocket?
(c) How much time will it take for the rocket to hit the ground?
5. Write an equation of a quadratic function whose graph lies in exactly three quadrants.
6. (a) How many $x$-intercepts could a quadratic function have?
(b) Give an equation of a quadratic having $k x$-intercepts for each $k$ you specified in part (a).
7. Find a quadratic function that is its own reflection in the vertical axis, or explain why none exists.
8. Find a quadratic function that is its own reflection in the horizontal axis, or explain why none exists.
9. What is the minimum number of $x$-intercepts a cubic function can have? What is the maximum number? Sketch a graph representing each possible scenario.
10. What is the minimum number of $x$-intercepts a degree 4 polynomial can have? What is the maximum number? Sketch a graph representing each possible scenario.
11. (a) Consider the power function $y=a x^{2}, a>0$. Explain (possibly using a sketch) how the shape of the function changes when the coefficient $a$ increases or decreases.
(b) Consider the power function $y=a x^{n}, a>0$. Explain (possibly using a sketch) how the shape of the function changes when the coefficient $a$ increases or decreases for fixed $n$. How is this change in shape different from the shape change that results from changing the power $n$ ?
(c) What happens for negative values of $a$ ?
12. Consider the two functions $f(x)=3 x^{2}$ and $g(x)=2 x^{5}$. Find all points of intersection of these functions and sketch them on the same set of axis. Repeat for functions $f(x)=x^{3}$ and $g(x)=4 x^{5}$. Repeat for functions $f(x)=5 x^{2}$ and $g(x)=3 x^{4}$.
13. Consider the functions $y=x^{n}, y=x^{1 / n}, y=x^{-n}$, where $n$ is an integer $n=1,2, \ldots$
(a) Which of these functions increases most steeply for values of $x$ greater than 1 ?
(b) Which decreases for large values of $x$ ?
(c) Which of these functions are not defined at $x=0$ ? Which functions are not defined for negative $x$ values?
(d) Compare the values of these functions for $0<x<1$.

Quadratic Functions: fill in the blanks to complete each row

| Graph(s) | Equation in the form $y=a x^{2}+b x+c$ | Equation in the form $y=a(x-h)^{2}+k$ | Axis of Symm. | $x$-ints. | Max/Min |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $y=x^{2}+2 x+4$ |  |  |  |  |
|  |  |  |  |  |  |
|  |  |  |  |  |  |
|  |  |  |  | $\begin{aligned} & x=1+\sqrt{2}, \\ & x=1-\sqrt{2} \end{aligned}$ |  |

Polynomial Functions: fill in the blanks to complete each row

| Graph(s) | Equation | $x$-intercepts | $y$-intercepts | Symmetry <br> (if any) |
| :---: | :---: | :---: | :---: | :---: |
|  |  | Exactly 2: $x=1+$ $\sqrt{2}$ and $x=1-\sqrt{2}$ |  |  |
|  | $y=x^{3}-x^{2}$ |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  | $y=x^{5}-2 x^{3}-6 x$ |  |  |  |

## Video for Module 4, part c: rational functions

## Rational functions

Rational functions are functions of the form $f(x)=\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$. Consider the function $f(x)=\frac{1}{x}$.

What is the domain of $f(x)$ ?

What are the $x$ and $y$ intercepts?

End behaviour: What happens to $f(x)$ as $x$ approaches infinity? negative infinity?

Sketch the graph of $f(x)$. Sketch the graphs of $g(x)=\frac{1}{x-2}$ and $h(x)=\frac{1}{x}-2$.

Sketch $f(x)=\frac{x^{2}-4}{x-2}$ for $x \geq 0$.
Sketch $f(x)=\frac{x-2}{x^{2}-4}$ for $x \geq 0$.

Consider $p(x)=\frac{x^{3}+7 x^{2}+6 x}{x^{2}-4}, q(x)=\frac{x-1}{x^{2}+4 x-5}$, and $s(x)=\frac{3 x^{2}}{x^{2}+1}$.
Find the domains of each $p(x), q(x), s(x)$.

What are the $x$ and $y$ intercepts?

Do they have any vertical asymptotes? If so, what are they?

Do they have any horizontal asymptotes? If so, what are they?

Sketch $f(x)=\frac{3 x^{2}}{x^{2}+1}$ for $x \geq 0$.

Vertical asymptotes of rational functions:

Horizontal asymptotes of rational functions:

Exercises.

1. Determine the equations of the vertical and horizontal asymptotes (if any) for the following functions:
(a) $f(x)=\frac{1}{x-12}$
(e) $m(x)=\frac{x+2}{x^{2}-4}$
(b) $g(x)=\frac{-x}{x^{2}-9}$
(c) $h(x)=\frac{x^{2}-2 x+1}{x}$
(f) $n(x)=\frac{x^{3}-8}{x^{2}}$
(d) $k(x)=\frac{x+2}{x^{2}+2}$
(g) $s(x)=\frac{4 x^{2}-1}{3 x^{2}+1}$
2. As a skydiver falls, his velocity keeps increasing. However, because of air resistance, the rate at which the velocity increases keeps decreasing and there is a limiting velocity that the skydiver cannot exceed. To see this behaviour, graph: $V=\frac{1000 t}{5 t+8}$, where $V$ is the velocity in feet per second and $t$ is the time in seconds. You can consider the domain of this function to be $[0, \infty)$ since time can not be negative.
(a) What is the horizontal asymptote for this graph?
(b) What is the limiting velocity that cannot be exceeded?
3. Determine the domain of the following:
(a) $F(x)=\frac{x}{x-3}$
(c) $H(x)=\sqrt{x-5}$
(b) $G(x)=\frac{x-2}{x^{2}-4}$
(d) $K(x)=\frac{-2}{\sqrt{x-5}}$
4. Which of the functions in Question 3 have vertical asymptotes? What are they?
5. Sketch the following rational functions:
(a) $f(x)=\frac{1}{x^{2}-4}$
(c) $h(x)=\frac{x+3}{x^{2}-9}$
(b) $g(x)=\frac{-1}{(x+1)^{2}}$
(d) $k(x)=\frac{x^{2}-2 x+1}{x-1}$

## 5 Exponential and logarithmic functions

- definitions and examples
- graphs: domain and range, intercepts, asymptotes
- solving exponential and logarithmic equations and inequalities
- growth/decay applications

Video for Module 5, part a: exponential and logarithmic functions
Exponential functions
Exponential functions are of the form $y=a^{x}$, where $a$ is a positive number.
Sketch $y=2^{x}$ and $y=2^{-x}$.

Sketch the graphs of $3^{x}, 5^{x}, 4^{-x},-4^{x}, e^{x}$.

Properties of exponential functions

Let $f(x)=2^{x+3}-5$. Find its domain and range. Sketch $y=f(x)$.

Solve the following equations:

$$
2^{x}=8^{2}
$$

$$
x^{2} e^{x}-2 x e^{x}=0
$$

$$
\left(\frac{1}{2}\right)^{x+4}=8^{x+1}
$$

$$
e^{2 x}-5 e^{x}+6=0
$$

For what values of $x$ is $e^{-x+2}>0$ ?

## $\underline{\text { Logarithmic functions }}$

Inverse of exponential functions is logarithmic function.
$y=2^{x}$
$y=\log _{2} x$

Solve the following equations:
$\log _{2} 8=$
$\log _{4} 2=$
$\log _{3} 1=$
$\log _{4} \frac{1}{4}=$
$\log _{3} 0=$
$\log _{2} 5 \frac{1}{5}=$

Fill in the blanks in the table below.

| Exponent form | Logarithm Form | Log used as an exponent |
| :---: | :---: | :---: |
| $2^{3}=8$ | $\log _{2} 8=3$ | $2^{\log _{2} 8}=8$ |
| $2-=1 / 4$ |  |  |
|  | $\log _{2} \ldots=4$ |  |
|  |  | $3^{\log _{3} 81}=$ |
| $4^{-3}=$ | $\ln 1=$ |  |
|  |  |  |
| $-3^{2}=$ |  |  |
| $(-3)^{2}=$ |  |  |
| $3^{-2}=$ |  |  |

Sketch $e^{x}$ and $\ln x$.
Sketch $e^{-x}$ and its inverse.

Let $f(x)=4-\ln (x+6)$. Find its domain and range. Sketch $y=f(x)$.

Solve the following equations:

$$
\log _{\frac{1}{3}} x=9
$$

Which is larger: $\log _{5} 100$ or $\log _{6} 100 ?$

$$
\log _{(1-x)} 16=2
$$

Which is larger: $\log _{10} 99$ or $\log _{9} 82$ ?

For what values of $x$ is $\ln (x-2)>0$ ?
Which is larger: $\log _{4} \frac{1}{18}$ or $\log _{4} \frac{1}{20}$ ?

Cancellation Laws: $e^{\ln x}=x \quad$ and $\quad \ln \left(e^{x}\right)=x$.

Laws of exponents and logarithms:

$$
\begin{array}{ll}
a^{x} \cdot a^{y}=a^{x+y} & \ln (x y)=\ln x+\ln y \\
\left(a^{x}\right)^{y}=a^{x y} & \ln \left(x^{p}\right)=p \ln x \\
a^{-x}=\frac{1}{a^{x}} & \ln \left(\frac{1}{x}\right)=-\ln x \\
\frac{a^{x}}{a^{y}}=a^{x-y} & \ln \left(\frac{x}{y}\right)=\ln x-\ln y
\end{array}
$$

Solve the following equations:

$$
e^{3 x}=5 \quad \ln x+\ln (x-5)=1
$$

$$
\ln (3 x)=5
$$

$$
\ln x+\ln x^{2}=5
$$

The magnitude on the Richter scale of an earthquake of intensity $I$ is given by $R=\log _{10} \frac{I}{I_{0}}$, where $I_{0}$ is the intensity of a "barely felt" earthquake, the so-called threshold.

1. If an earthquake is 10,000 times as intense as threshold, what is its magnitude on the Richter scale?
2. The strongest earthquake on record happened in Chile on May 22, 1960 with the magnitude 9.5 . How much stronger was this earthquake than the threshold?

The simplest growth and decay model is $P(t)=P_{0} e^{k t}$.

Two populations of fish in Cultus Lake are studied. Sturgeon population is modelled by $S(t)=S_{0} e^{-1.3 t}$ and rainbow population by $R(t)=R_{0} e^{1.7 t}$, where $t$ is time in years.
a) Are the populations growing or declining?
b) If the initial levels of the two populations were 500 and 300 respectively, at what time would the two populations be exactly equal?
c) On the same set of axis, sketch the graphs of $S(t)$ and $R(t)$.

The half-life of Cesium-137 in laboratory conditions is known to be 30 years.
a) How much of the initial amount of Cesium-137 does the soil in the areas affected by the accident in 1986 still contain?
b) Data shows that the amount of Cesium-137 in soil near Chernobyl is not decreasing as it does in the lab. Find the ecological half-life if the soil in the areas affected by the accident in 1986 still contains $90.48 \%$ of the initial amount of Cesium-137.
c) Assuming ecological constants, in how many year will there be $5 \%$ of the initial amount of Cesium- 137 left?
d) Sketch the graph describing the ecological decay of Cesium-137.

Paul's chickens got the bird flu and he is trying to stop the epidemic. He knows that the function

$$
f(t)=\frac{300}{5+20 e^{-0.07 t}}
$$

describes the number of his chickens who are sick $t$ weeks after the initial outbreak.
a) How many chickens became sick when the flu epidemic has began?
b) In how many weeks will 30 chickens be sick?
c) What is the maximum number of chickens that will become ill?
d) Sketch the rough graph of the epidemic spread.

## Exercises

1. See table on the next page. Fill in all the blank cells.
2. Use graphs to justify why the equation $x=\ln x$ has no solutions.
3. Write each of the following as a sum or difference of logarithms. Simplify and eliminate exponents as far as possible.
(a) $\ln \left[\frac{e x^{3}}{(x+1)^{2} \sqrt{x-1}}\right]$
(b) $\ln \left[\frac{(x+1) \sqrt{3 x-2}}{x^{2}(2 x-1)^{4}}\right]$
(c) $\ln \left(\frac{x^{2} \cdot y}{e^{z}}\right)$
4. Simplify, if possible:
(a) $\ln (\ln e)+\ln \left(e^{2} \cdot e^{5}\right)$
(b) $\frac{x^{2}-\log _{3} 81}{x \ln e+3}$
5. Find some values of $a$ and $b$ so that $\log _{2}(a+b)$ is not equal to $\log _{2} a+\log _{2} b$. Do not use a calculator; instead, try some values of $a$ and $b$ where the expressions are easy to calculate.
6. Solve for $x$ in each of the following.
(a) $2^{x^{2}}=8^{2}$
(e) $\log _{6} x+\log _{6}(x-5)=1$
(i) $10^{\sin x}=1$
(b) $8^{x-7}-1=0$
(f) $e^{4 x}+7 e^{2 x}-2=0$
(c) $3^{2 x+1}=7$
(g) $(\ln x)^{2}=\ln x^{2}$
(j) $\ln x-\ln \sqrt{x}=\frac{1}{2}$
(d) $(1.025)^{12 t}=3$
(h) $2 x e^{x}+x^{2} e^{x} \geq 0$
(k) $\ln x+\ln (x-1)=\ln (x+2)$
7. Solve the following inequalities:
(a) $x^{2} e^{x}-2 x e^{x} \geq 0$
(b) $\log _{3}\left(x^{2}+2 x+1\right)>0$
(c) $\frac{x+1}{e^{x}} \geq 0$
8. The magnitude on the Richter scale of an earthquake of intensity $I$ is given by $R=\log _{10} \frac{I}{I_{0}}$, where $I_{0}$ is the intensity of a "barely felt" earthquake, the so-called threshold. In February 2016, Vancouver got hit by an earthquake. Natural Resources Canada initially measured it at 4.3 magnitude, while the U.S. Geological Survey reported 4.8. Compare the intensity of the original Canadian measure to the intensity of the American one; that is how much stronger is the 4.8 quake than 4.3 quake?
9. Each bacteria divides once a minute so that the total number of bacteria doubles every minute. At 1 pm , you put one bacteria in the glass and at 2 pm the glass is full. At what time was the glass half full?
10. Iodine 131 has a half life of about 8 years. Assuming that you started with 100 ml , sketch the graph representing the decay of Iodine 131 . What happens to the Iodine as time goes on?
11. A population of 20 bacteria doubles every 12 hours. Sketch the graph describing the population. What is the formula that describes the population growth? What happens to the population as time goes on?
12. A population of 20 bacteria increases to 30 after 12 hours. Assuming the growth is exponential, how long will it take for the population to double? Sketch the graph of the population growth. What happens to the population as time goes on?
13. According to the bunny specialists of the UVic, in 1990 there were 224 bunnies living on campus. By 2000, the number had increased to 453.
(a) Assuming the population grows according to the model $P(t)=P_{0} e^{k t}$, find the growth equation and the growth constant $k$. Use the resulting model to project bunny population in 2020.
(b) In which year will the bunny population reach 1000 ?
(c) In which year will the bunny population double?

Exponential and Logarithmic Functions: fill in the blanks to complete each row

| Graph(s) | Equation | Domain | Range | $x$-int | $y$-int | Asymptote |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $y=3^{x}$ |  |  |  |  |  |
|  | $y=\left(\frac{1}{3}\right)^{x}$ |  |  |  |  |  |
|  | $y=\log _{2} x$ |  |  |  |  |  |
|  | $y=\ln (x-3)$ |  |  |  |  |  |
|  | $y=2^{x}+2$ |  |  |  |  |  |

## 6 Trigonometry

- unit circle, angles, radians and degrees
- definitions of trigonometric functions
- graphs of trigonometric functions

Video for Module 6, part a: introduction to trig functions
Radians and degrees:

- triangle trigonometry
- trigonometric identities
- trigonometric equations



Definition of sine and cosine functions



$$
\begin{array}{ll}
\cos 0= & \cos (\pi / 2)= \\
\sin 0= & \cos \pi=
\end{array}
$$

$\cos (3 \pi / 2)=$
$\cos (7 \pi)=$
$\sin (-5 \pi / 2)=$
$\sin (\pi / 2+2 k \pi)=$

Answer the following questions using unit circle definitions of sine and cosine:
How many solutions are there to the equation $\sin \theta=0.7$ ?

Evaluate $\sin (k \pi)$., where $k$ is an integer.

Which of the following expressions are true, if any?

$$
\begin{aligned}
& \cos (-\theta)=\cos (\theta) \\
& \sin (-\theta)=\sin (\theta) \\
& \sin (\theta+\pi)=\sin \theta \\
& \cos 2 \theta=2 \cos \theta
\end{aligned}
$$

Special angles and special triangles

$\cos (3 \pi / 4)=$

$\sin (7 \pi / 4)=$

$\cos (-\pi / 6)=$

$\sin (2 \pi / 3)=$


Graphs of sine and cosine




Adding various graph transformation, the general form of a sine function is $y=a+b \sin (c(x+d))$.


$y=\sin (x / 3)$





Suppose $\theta$ is an angle in 4 th quadrant such that $\cos \theta=\frac{2}{5}$. Find the values of $\sin \theta, \tan \theta$ and $\sec \theta$.


$$
\tan \frac{5 \pi}{4}=
$$

$\tan x \cdot \cos x=$

$$
\begin{aligned}
& \sec \frac{\pi}{6}= \\
& \frac{\sec x}{\tan x}=
\end{aligned}
$$

If the secant of the angle $\theta$ is -3 , what is the secant of $-\theta$ ?

## Graphs of other trigonometric functions

Tangent


Variations of tangent: $y=a+b \tan (c(x+d))$

Secant


Variations of secant: $y=a+b \sec (c(x+d))$

## Exercises.

1. Without a calculator, determine which one is larger:
(a) $\sin 1^{\circ}$ or $\sin 1 ?$
(b) $\sin 13^{\circ}$ or $\cos 13^{\circ}$ ?
(c) $\tan 23^{\circ}$ or $\sin 23^{\circ}$ ?
2. Without using a calculator, arrange the following values from smallest to largest (note, some of these might be equal to each other):

$$
\sin \frac{\pi}{3}, \cos \frac{\pi}{3}, \cos \frac{2 \pi}{3}, \sin \frac{-\pi}{3}, \cos \frac{4 \pi}{3}, \sin -\frac{2 \pi}{3}
$$

3. Determine how many roots do the following equations have for $\theta \in[0,2 \pi)$ (do NOT solve them):
(a) $\sin \theta=1$
(c) $\cos 2 \theta=\frac{1}{3}$
(b) $\tan \theta=0$
(d) $\cos 3 \theta=\frac{1}{3}$
4. If the tangent of an angle is negative and its secant is positive, in which quadrant does the angle terminate?
5. Assuming $k$ is a whole number, simplify each of the following:
(a) $\sin (\pi / 2+2 k \pi)$
(c) $\cos (-\pi / 2+2 k \pi)$
(e) $\sin (\pi / 2+k \pi)$
(b) $\sin (k \pi)$
(d) $\cos (k \pi)$
(f) $\cos (-\pi / 2+k \pi)$
6. Graph each of the following:
(a) $\sin (x-\pi)$
(b) $\sin (x+\pi / 4)$
(c) $3 \tan (x-\pi)$
(d) $-\sec (x-\pi / 2)$
7. Recall the definition of $\sin \theta$ and $\cos \theta$ as $y$ and $x$ coordinates of the point moving along a unit circle. What if instead we used a unit square?


Let's consider the functions corresponding to vertical and horizontal movements; call them squine and cosquine. Follow the similar idea from the handout in class to draw the graphs of squine and cosquine.
8. Determine the formulas for the cosine function in part a and the sine function in part b:

(a)
(b)


## Video for Module 6, part b: triangle trigonometry

$\underline{\text { Triangle trigonometry }}$


Given the triangle below, find the values of $\sin \alpha, \tan \alpha$ and $\sec \alpha$.


Find the unknown sides of the triangle in the picture below:


To find the height of a tree, a person walks to a point 30 feet from the base of the tree. She measures an angle of 57 degrees between a line of sight to the top of the tree and the ground, as shown in the picture below. Find the height of the tree.


## Exercises.

1. Consider the following triangle:

(a) If $a=4$ and $c=5$, find the rest of the sides and sines of all angles.
(b) If $B=30^{\circ}$ and $a=6$ find the rest of the sides and cosines of all angles.
(c) If $B=60^{\circ}$ and $b=6$ find the rest of the sides and tangents of all angles.
(d) If $B=45^{\circ}$ and $a=1$, find the rest of the sides and all the angles.
2. In each of the following, find $x$ :

3. You are climbing the Abby Grind and you're facing your last peak. You'd like to know how high it is, so you measure the angle of elevation from two points 1500 feet apart. The angles are 30 and 35 degrees.


Find the height of the peak.
4. There is an antenna on the top of a building. From a location 300 feet from the base of the building, the angle of elevation to the top of the building is measured to be 40 degrees. From the same location, the angle of elevation to the top of the antenna is measured to be 43 degrees. Find the height of the antenna.
5. Suppose a straight line makes an angle of $\alpha$ with the positive $x$-axis and has $y$-intercept of $k$. Find the equations of this line. (Hint: what trigonometric function equals to the line's slope?)
6. Many of the examples and graphs in this section are from https://philschatz.com/precalculus-book/ contents/m49384.html. Visit the website for more examples and exercises.

Video for Module 6, part c: inverse trigonometric functions
$\underline{\text { Inverse trigonometric functions }}$

Start with $y=\sin x$ :


Restrict the domain to $x \in\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ :


Properties of $\sin ^{-1} x$

Evaluate, where possible:

$$
\sin ^{-1} 0=
$$

$$
\sin ^{-1} 1=
$$

$$
\sin ^{-1} \frac{1}{2}=
$$

$\arcsin \pi=$

Start with $y=\cos x$ :


Restrict the domain to $x \in[0, \pi]$ :
Properties of $\cos ^{-1} x$


Consider $y=\tan x$


Restrict the domain to $x \in\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$


Evaluate, where possible:

$$
\cos ^{-1}(-1)=\quad \tan ^{-1} 1=
$$

$$
\cos ^{-1}(3)=
$$

$$
\tan ^{-1}(3)=
$$

Re-write using inverse trig functions:

$$
\cos (\pi / 3)=1 / 2 \quad \cos (1 / 2) \approx 0.8776 \quad \tan (\pi / 4)=1 \quad \sin \theta=0.3
$$

Find $\theta$ in the following triangle:


Basic inverse laws:

$$
\begin{array}{lll}
\sin \left(\sin ^{-1} x\right)=x \text { for } x \in & \cos \left(\cos ^{-1} x\right)=x \text { for } x \in & \tan \left(\tan ^{-1} x\right)=x \text { for } x \in \\
\sin ^{-1}(\sin x)=x \text { for } x \in & \cos ^{-1}(\cos x)=x \text { for } x \in & \tan ^{-1}(\tan x)=x \text { for } x \in
\end{array}
$$

Evaluate:

$$
\begin{array}{ll}
\sin \left(\sin ^{-1}(1 / 2)\right)= & \sin ^{-1}(\sin \pi / 4)= \\
& \\
\sin \left(\sin ^{-1}(2)\right)= & \sin ^{-1}(\sin (2 \pi))=
\end{array}
$$

$\tan ^{-1}(\tan (2))=$

Simplify $\sin \left(\cos ^{-1}(3 / 5)\right)$.

Simplify $\tan \left(\sin ^{-1} x\right)$.

Solving trigonometric equations

Solve $\sin x=2 / 3$.

Solve $3(\tan x+1)=2 \tan x$.

Solve $2 \sin ^{2} x-1=0,0 \leq x<2 \pi$.

Solve $2 \cos x+1=2$ for $x \in[0,2 \pi)$.

Solve $\sec x=3$ for $0 \leq x<2 \pi$.

Solve $2 \cos ^{2} x+\cos x=0$.

Solve $\sin ^{2} x-3 \sin x+2=0,0 \leq x<2 \pi$.

Solve $\sin ^{2} x=2 \cos x+2$ for $x \in[0,2 \pi)$.

Solve $9-\sin ^{2} x=0$ for $x \in[0,2 \pi)$.

Solve $\cos 2 x=\cos x$.

Solve $\sin (2 x)=\cos (2 x), 0 \leq x<2 \pi$.

## Exercises.

1. Explain why $\frac{\sin x+\tan x}{\cos x+\cot x}$ cannot be negative.
2. What is $\frac{\sin x+\cos x}{\sin x-\cos x}$ if $\quad$ a) $\sin x \cdot \cos x=0.4$ ? $\quad$ b) $\tan x=3$ ?
3. Simplify the following expressions:
(a) $\frac{1-\sin x}{\cos x}-\frac{\cos x}{1+\sin x}$
(b) $\frac{1}{1+\tan ^{2} x}+\frac{1}{1+\cot ^{2} x}$
4. Solve the following equations:
(a) $\sin x=\frac{1}{2}$
(c) $\sin (x+\pi / 4)=\frac{1}{2}$
(e) $\cos x=-\frac{1}{2}$
(g) $\cos ^{2} x-\cos x-2=0$
(b) $\sin 2 x=\frac{1}{2}$
(d) $\tan x+\cot x=\frac{1}{2}$
(f) $\cos ^{2} x-\cos x=0$
5. Several examples in this section are from Sections 7.1 and 7.5 of OpenStax Precalculus book available at https://philschatz.com/precalculus-book/. Visit the website for more examples and exercises.
6. Find the exact values of the following. Show your work, do not use a calculator.
(a) $\frac{\sin \left(\frac{5 \pi}{4}\right)}{1-\cos \left(\frac{5 \pi}{4}\right)}$
(b) $\sin \left(\frac{19 \pi}{2}\right)+\sin \left(-\frac{19 \pi}{2}\right)$
(c) $\frac{\cos \left(\frac{19 \pi}{3}\right)}{\cos \left(\frac{23 \pi}{3}\right)}$
7. Given that $\sin t=-\frac{2}{3}$, find all possible values of $\cos t$.
8. Given that $\cos t=\frac{1}{5}$ and $t$ is an angle in the fourth quadrant, find $\sin t$.
9. Sketch at least one cycle of each of the given functions. In each case, state the period as well as the range of the function.
(a) $f(x)=1+\sin \left(2 x-\frac{\pi}{3}\right)$
(b) $f(x)=\cot \left(3 x+\frac{\pi}{6}\right)$.
10. Evaluate each of the following without using a calculator. Give an exact answer, using radicals if needed.
(a) $\arcsin (\sqrt{3} / 2)$
(b) $\tan (\arcsin (-1 / 2))$
11. The function

$$
f(x)=98.6+0.3 \sin \left(\frac{\pi}{12} x-\frac{11 \pi}{12}\right)
$$

models variation in body temperature (in Fahrenheit) $x$ hours after midnight.
(a) What is the body temperature at midnight?
(b) What is the period of the body temperature cycle?
(c) When is the body temperature highest and what is it at that time?
(d) When is the body temperature lowest and what is it at that time?
(e) Sketch one period of the function.

## 7 Applications

- setting up algebraic expressions and equations in word problems
- working with various shapes (rectangles, circles, spheres, cylinders)
- right angle triangles
- trigonometric word problems

Video for Module 7: applications
General approach to setting up words problems:

- sketch a diagram
- identify all the given quantities, assign variables
- figure out equation(s) connecting your variables
- set up the necessary function, if necessary eliminate all but one variable

Suppose the perimeter of a rectangular field is 100 metres and that the length of the field is six times larger than the width. What are the dimensions of the field?

Paul wants to build a rectangular pen for his chickens alongside his house. Paul only has 30 meters of fencing available. Set up the function representing the pen's area in terms of only one variable.

Seth has 100 meters of fencing to built two pens: a square one for his pygmy goats and a circular one for his miniature pigs. Set up the function for the total combined area of the pens in terms of only one variable.

A Norman window has the shape of two squares, one of top of the other, surmounted by a semicircle. Express the perimeter and area of the window as a function of its width.

You want to make a cylindrical can of volume 100 cubic cm. Express the surface area (not including the top and bottom) as the function of a radius.

A open box is constructed from a rectangular 8 by 10 sheet of cardboard by cutting out identical squares of side $x$ out of each corner and folding up the edges. Find the volume of the box in terms of $x$.

Shane is building a rain shelter to store 250 cubic meters of lumber over the winter. The shelter is to have a back, two square sides and a top. The sides of the shelter will all be made out of plastic that costs $\$ 5$ per square meters, while the metal material for the roof costs $\$ 10$ per square meter. Express the cost of the shelter in terms of one variable.

A 30-foot ladder is placed against the wall, with its bottom $x$ feet away from the wall. Find the vertical height $h$ along the wall from the ground to the of the top of the ladder in terms of $x$.

A 6 -foot man is $x$ feet away from a 15 -foot streetlight. Find the length $L$ of his shadow as a function of $x$.

A rocket flies off vertically from its base and is being following by the camera set up on the ground 2 km away from the launching site. Find a relation between the height of the rocket and the angle of elevation of the camera.

A crocodile is stalking a zebra located 20 meters upstream on the opposite side of the river of width 5 meters. The crocodile can swim at 1 meter per second and run at 60 cm per second on land. Assume the zebra doesn't move. Express the time it will take the crocodile to get to zebra as a function of $x$ as labelled in the picture.


## Exercises.

1. Shane has 76 meters of fencing and he constructs a rectangular pen for his dogs that is 2 meters longer than it is wide. What is the area enclosed by this fence?
2. Kaylin and Adam need to build a gate for their llamas enclosure. The width of the gate is to be 2 feet more than its heigh and the diagonal board they are planning to use for bracing is 8 ft long. What are the dimensions of the gate they can build?
3. Paul has recently decided to add ducks and turkeys to his chicken colony. Unfortunately, the three species of birds do not get along. Paul has 80 meters of fencing to enclose a rectangular coop and put two parallel fences across it to divide it into 3 smaller coops, all in the effort to prevent potential bird fights. Set up the function representing the coop's area in terms of only one variable.
4. The surface area of a sphere is $S(r)=4 \pi r^{2}$ and the volume of a sphere is $V(r)=\frac{4}{3} \pi r^{3}$. Express volume as a function of surface area and vice versa.
5. Shane wants to plant an apple orchard. From past experience, he knows that if he plants 16 apple trees, they will yield on average 80 apples per tree. However, once the garden becomes crowded with trees, they yield fewer fruit. In fact, for every additional tree planted on top of 16 , the yield will decrease by 4 apples per tree. Find the equation for the total yield in terms of the number of trees planted.
6. Kaylin's grain silo has a shape of a cylinder with radius $r$ and height $h$ with a hemispherical top. Express the heigh of the silo as a function of the radius if the total volume of the silo is 1000 cubic meters.
7. Some airlines have restrictions on the size of items of luggage that passengers are allowed to take with them. Suppose that one has a rule that the sum of the length, width and height of any piece of luggage must be less than or equal to 210 cm . A passenger wants to take a box of the maximum allowable volume. If the length and width are to be equal, express the volume in terms of length.
8. Amanda wants to print a poster-sized picture of her dog Chrissy. A printer tells her that she can order a poster of any dimensions but its total area cannot exceed 1000 square centimetres. Moreover, the printing machine requires the poster to have 5 cm margins on each side and 7 cm margins on top and bottom of the poster. Find an expression for the printed area in terms of one variable.
9. A piece of wire 80 inches long is cut into at most two pieces. Each piece is bent into the shape of a square. Express the total enclosed area in terms of one variable.
10. Kseniya buys a dog house for Schumi and has to get it delivered to her house. Her total bill includes the price of the dog house, $7 \%$ tax and the delivery fee of $\$ 50$.
(a) Write a function $t(x)$ representing Kseniya's total bill after taxes on the purchase amount $x$.
(b) Write a function $f(x)$ for the total bill after taxes and delivery fee on the purchase amount $x$.
(c) Calculate and interpret $t(f(x))$ and $f(t(x))$. Which will result in lower cost for Kseniya?
(d) If taxes cannot be charges on delivery fees, which function $t(f(x))$ or $f(t(x))$ must be used?
11. The Greek mathematicians measured the diameter of the Earth long before it was a well-accepted fact that the Earth was round. We will now use their method. One day, I am standing near the equator, where I notice that the Sun is directly above me at noon. The next day, I move to a city 500 km away, where I measure (with a simple sundial) the angle of sun rays to be at $4.92^{\circ}$.


Estimate the diameter of the Earth.

