Vector Consulting

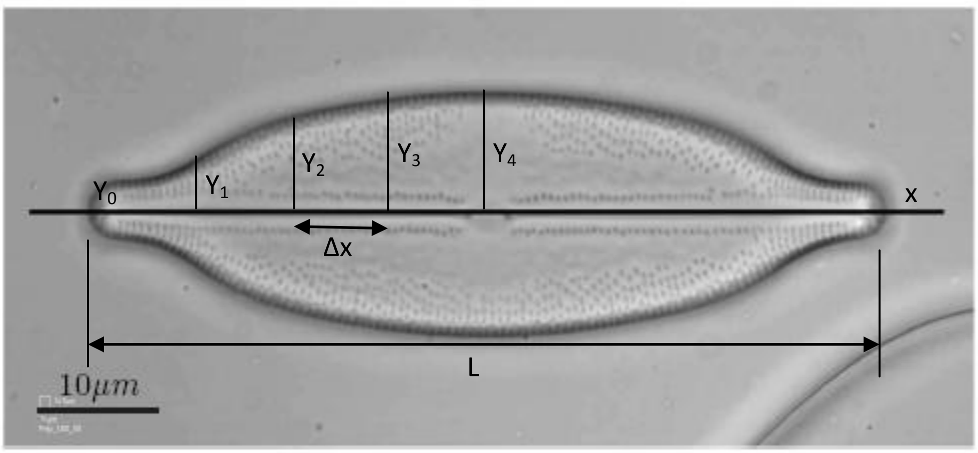
To: Henry Waters
From: Ian Rennie

Problem 1: Biostar International is studying metabolic needs and growth rates of diatoms that they wish to grow in their tanks. Towards that they need some way to estimate the body volume of individual diatoms, as accurately as possible, but without too much effort. They are now concentrating on "pinnate diatoms" which have radial symmetry: a microscope image of a typical specimen is shown below, but the dimensions can vary between individuals. Their microscopes have installed micro-callipers which they will use to make the measurements. Along with wanting to make as few measurements as possible they would like their estimates to be within 5% of the actual volume.

The approximate volume (V), in micrometers, of a typical diatom was found to be,

$$V \cong \frac{L*\pi}{12}*(y_0^2+4y_1^2+2y_2^2+4y_3^2+y_4^2)$$

Where (L) is the length of the diatom, in micrometers and each corresponding (y_n) value is a radius as shown below, in micrometers.



The volume equation was derived by using the area approximation method of Simpson's rule,

$$\int_a^b f(x)\,dx \cong S_n = \left(\frac{\Delta x}{3}\right)(y_0+4y_1+2y_2+4y_3+2y_4+\cdots 2y_{n-2}+4y_{n-1}+y_n)$$

Where Δx=(b-a)/n and y=f(x_i) and x_o,x_1,...x_n partition [a,b] with n and even number.

In order to measure the diatom, 5 radial measurements will be made starting at one end and taking the measurements at evenly spaced intervals until the half way point (L/2) of the diatom. So the area approximation of the quarter of the diatom shown becomes,

$$\int_0^{L/2} f(x)\,dx \cong S_4 = \left(\frac{\Delta x}{3}\right)(y_0+4y_1+2y_2+4y_3+y_4)$$

Where f(x) is the curve shown graphically above (outer edge of the top of the diatom), and (S_4) is the area approximation using Simpsons rule with 4 subintervals. (Δx) is the total length of the subintervals (L/2) divided by the number of subintervals (N=4). Therefore the equation becomes

$$\int_0^{L/2} f(x)\,dx \cong S_4 = \left(\frac{\mathrm{L}}{3*2*4}\right)(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)$$

Assuming the diatom is symmetric left to right then the total area of the top half will double,

$$\int_0^{L} f(x)\,dx \cong S_4 = \left(\frac{\mathrm{L}}{3*2*4}\right)(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) * 2$$

To find the volume of the diatom the area found can be rotated around the x-axis to make a cylindrical solid.

The rotation formula is,

$$\pi * \int_a^b (f(x))^2\,dx$$

Where (a) and (b) represents the interval of rotation f(x) is the function evaluated at an (x) value.
The diatom's (y) values are also a function evaluated at a specific (x) value so the relationship between the two equations can be made by squaring all the individual terms and multiplying the equation by (π). Therefore, the resulting equation will become the approximate volume,

$$\boxed{V \cong \frac{L*\pi}{12} * (y_0^2 + 4y_1^2 + 2y_2^2 + 4y_3^2 + y_4^2)}$$

This equation only uses five radius measurements so it appears to result in a very approximate volume value. Using an above style of equation the volume was calculated with five radius measurements and also forty-seven radius measurements. The following values were found,

$$S_4 = 12086.51638\,\mu m^3$$

$$S_{46} = 12517.58065 \mu m^3$$

The volume calculated with 47 measurements is a much more accurate measurement and differs by only 3.44% from the value calculated with five measurements. Therefore, I can confidently say that the above volume equation will give the volume of a diatom within 5%.

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Problem 2: Biostar is designing a dry-land holding-pen for salmon fry. The tank is going to be of volume 2 million litres. The water in the tank will have an initial charge of sea-water to introduce the fry. The water will run through a filtration system then back into the tank to remove impurities, but along with the impurities some of the salt content and the water itself will be removed. Therefore, the salinity of the tank will change over time. Below is some data they provided us:
-The tank has a total volume of 2 million litres.
-Sea-water in the area has a concentration of 25 grams per litre.
-Once the tank has been given its initial fill of sea-water, water will be pumped out at a rate of 100000 litres per day, sending it through a filter. The filter will remove impurities, but also about 40 percent of the salinity. About 10000 litres of water will be lost in the filtration process as well, leaving 90000 litres of filtered, somewhat desalinated water to return to the tank (each day). (So the 90000 litres will have a salinity of 60 percent of the current tank salinity).
-To keep the tank volume constant there will be 10000 litres of sea-water added each day.
They want us to come up with a formula that produces the salinity of the water with respect to the time (in days).

The formula is:

$$Q = 10869565.22 + 39130434.78e^{-.023t}$$

Where (Q) is the quantity of salt (in grams), and (t) is the time (in days).

This formula was derived using a differential equation. ($\frac{dQ}{dt}$) will be the rate of desalination represented by the equation,

$$\frac{dQ}{dt} = Rate\ in - Rate\ out$$

The rate in is the amount of salt coming into the tank on a daily basis and the rate out will be the quantity of salt leaving the tank per day.

The rate out will be the 100000 L of water exiting the tank as a result of the filtration process. This water will have the same concentration of salt in it as the tank so therefore,

$$Rate\ out = 100000\left(\frac{L}{day}\right) * \frac{Q}{2000000L}$$

Where (Q), in grams, is some function of the quantity of salt currently in the tank with respect to time over the total volume of the tank (to give concentration of the water).

The rate in will be the sum of the 10000 L of sea-water being pumped in each day to keep the volume constant and the 90000 L of partly desalinated water coming back into the tank. The sea-water has a concentration of 25g/L and the desalinated water has 60% of the quantity of salt still in the tank. Therefore,

$$Rate\ in = 10000\left(\frac{L}{day}\right) * 25\left(\frac{g}{L}\right) + 90000\left(\frac{L}{day}\right) * \frac{0.6Q}{2000000L}$$

Where (Q), in grams, is some function of the quantity of salt currently in the tank with respect to time over the total volume of the tank (to give concentration of the water).
Combing the three equations gives,

$$\frac{dQ}{dt} = 10000\left(\frac{L}{day}\right) * 25\left(\frac{g}{L}\right) + 90000\left(\frac{L}{day}\right) * \frac{0.6Q}{2000000L} - 100000\left(\frac{L}{day}\right) * \frac{Q}{2000000L}$$

Simplifying gives,

$$\frac{dQ}{dt} = 250000\left(\frac{g}{day}\right) - .023(day^{-1}) * Q$$

If $\frac{dy}{dx} = ay + b$ and (y) is some function of (x) then $y = -\frac{b}{a} + Ge^{ax}$ where (G) is some case specific constant. Therefore, the above equation becomes,

$$Q = 10869565.22 + Ge^{-.023t}$$

Where (t) is the time (in days) since the initial charge of sea water has been added. To find (G), (the case specific constant) the initial condition can be plugged into the equation. Initially the tank will contain only fresh sea-water with concentration of 25g/L. The tank is 2 million L so initially, at time t=0 days, there will be 50,000,000 g of salt in the tank. Therefore,

$$50000000(g) = 10869565.22(g) + Ge^{-.023(0\,days)}$$

Solving for (G),

$$G = 39130434.78\,(g)$$

Therefore the equation for the quantity of salt in the tank over a certain amount of time is,

$$\boxed{Q = 10869565.22 + 39130434.78e^{-.023t}}$$

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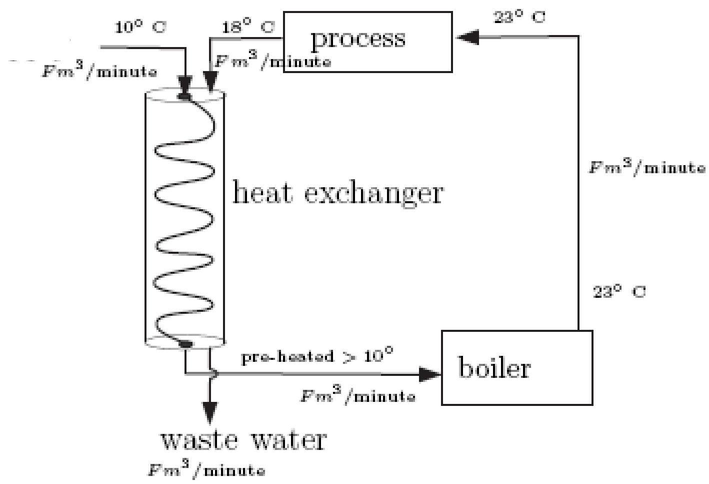
Problem 3: Biostar is hoping to decrease their energy consumption by decreasing the amount of energy they use heating water. They want to use the somewhat heated waste water to heat the uncontaminated water as much as they can before sending it through the boiler. The device they are going to use in this process is shown below.



The fresh water is typically a temperature of 10°C, they heat the water to 25°C, and the temperature after the process (waste water) is 18°C. As shown in the diagram the ex-process water forms a bath around the fresh water tube in turn pre-heating the water. This will result in less energy needed to heat the water to 23°C.
The longer the water stays in the heat exchanger the higher the resulting temperature of the water will be so the flow rate (F in m^3/min) will play a factor. The flow rate can only be between 0 and 0.02, the coil inside the exchanger is 100 metres long and 2cm in diameter. Also, if the temperature of the water bath and the temperature of the fresh water are within 1 degree then the temperature will be rising at a rate of 0.1^0/min.
For a batch size of $100m^3$ they need: a formula of their energy expenditure (in Joules) at any flow rate; a comparison of the energy expenditure with and without the heat exchanger; and a graph to display the data collected.

The formula for the heat expenditure was found to be,

$$q = \left(5 + 8 * e^{\left(-\frac{0.001*\pi}{F}\right)}\right) * 418600$$

Where (q) is the heat expenditure (in joules) and (F) is the flow rate (in m^3/min).

This equation was derived by starting with the relationship that it takes 4184 Joules of heat to raise one litre of water one degree Celsius. Therefore more generally,

$$q = \Delta T * 4184 * V$$

Where (q) is the heat expenditure (in joules), (ΔT) is the change is temperature (in °C), and (V) is the volume of water (in litres). In our case the volume will be 100 m^3 or 100 L.

Therefore the equation becomes,

$$q = \Delta T * 4184000$$

The change in temperature is the final temperature minus the initial temperature. The final temperature will be the heat of the water coming out of the boiler, which is 23°C, and the initial will be the temperature going into the boiler (or out of the heat exchanger). Now the equation becomes,

$$\boxed{q = (23 - T) * 4184000}$$

Where (T) is some function of the temperature with respect to the flow rate.

Now to derive the formula for (T), in degrees Celsius. Using Newton's law of heating, the rate at which the fresh water's temperature is changing is proportional to the difference between the fresh and waste water's temperatures. Therefore,

$$\frac{dT}{dt} = k(T_w - T)$$

Where ($\frac{dT}{dt}$) is the rate in which the fresh water's temperature is changing (in °C/min), (T_w) is the waste water's temperature (18°C), (T) is the fresh water's temperature which will be some temperature function with respect to time, and (k) is some proportionality constant. We know that when the temperature difference is 1°C then the rate of change will be 0.1°/min; therefore, when (T) is 17°C,

$$0.1 = k(18 - 17)$$

And solving for (k),

$$k = 0.1$$

Plugging the proportionality constant and (T_w) in the equation gives,

$$\frac{dT}{dt} = 0.1(18 - T_f)$$

And simplifying,

$$\frac{dT}{dt} = 1.8^o - 0.1 * T$$

Using the relationship, If $\frac{dy}{dx} = ay + b$ and (y) is some function of (x) then $y = -\frac{b}{a} + Ge^{ax}$ where (G) is some case specific constant. Then, the above equation becomes,

$$T = \frac{1.8}{0.1} + Ge^{-0.1*t}$$

Where (G) is some case specific constant and (t) is the time (in min). To find this constant we use the fact that the water entering the heat exchanger is 10°C. So, when t=0 min then temperature is 10°C. Which results in,

$$10 = 18 + Ge^{-0.1*0}$$

Solving for (G),

$$G = -8$$

So the formula becomes,

$$T = 18 - 8e^{-0.1*t}$$

The next relationship will be the relationship between the flow rate (F), in m^3/min, and the time spent in the exchanger (t), in minutes. The flow rate is the quantity of water travelling through the tank over some time interval.
Therefore,

$$F = V/t$$

Rearranging gives,

$$t = V/F$$

In this case the volume of water will be the contents of the tube which is the volume of the tube. The tube is a cylinder with a radius of 0.01m and a length of 100m. Using the equation for volume of a circular cylinder,

$$V = \pi * r^2 * l$$

And plugging in the values above,

$$V = \pi * 0.01^2 * 100$$

Solving,

$$V = \pi * 0.01$$

Now the equation becomes,

$$t = \frac{\pi * 0.01}{F}$$

And inserting into the temperature equation,

$$T = 18 - 8e^{-\left(\frac{\pi*0.001}{F}\right)}$$

And plugging into the heat equation,

$$q = \left(23 - \left(18 - 8e^{-\left(\frac{\pi*0.001}{F}\right)}\right)\right) * 4184000$$

Simplifying, results in the equation for heat expenditure with respect to the flow rate,

$$\boxed{q = \left(5 + 8 * e^{\left(-\frac{0.001*\pi}{F}\right)}\right) * 418600}$$

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To: Biostar International
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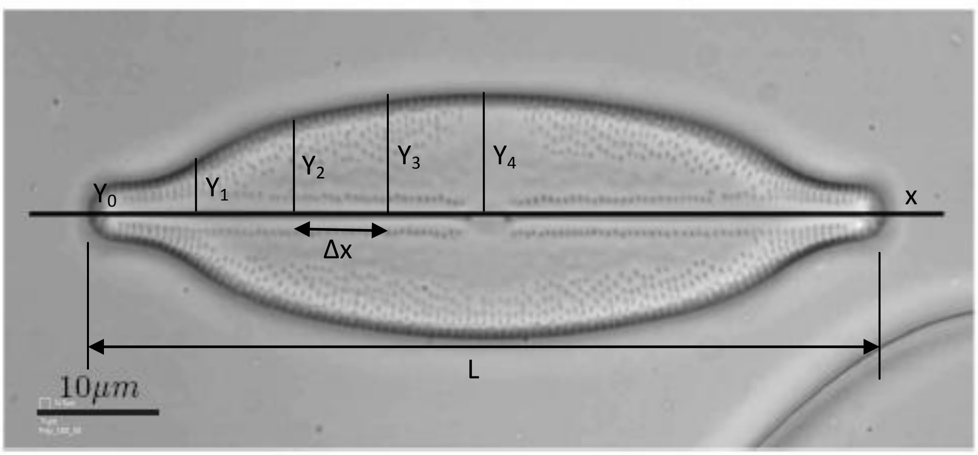
Problem 1: The equation for the volume of a typical diatom (V) was found to be,

$$V \cong \frac{L * \pi}{12} * (y_0^2 + 4y_1^2 + 2y_2^2 + 4y_3^2 + y_4^2)$$

Where (L) is the length of a diatom, in micrometers, and each (y_n) value is a radial measurement as shown in the diagram below. Each vertical measurement should be made at specific distances a part, (Δx). The value of (Δx) will be the Length divided by eight,

$$\Delta x = \frac{L}{8}$$



To use the equation make the measurements as shown above and simply plug them into the equation.

After analyzing the typical diatom presented to us and performing rigorous measurements and calculations we can confidently say that this equation will produce a value within 5% of the actual volume of a diatom.

Problem 2: The equation for the quantity of the salt (Q) in the tank, in grams, over a certain amount of time is,

$$Q = 10869565.22 + 39130434.78e^{-.023t}$$

Where (t) is the amount of time, in days, since the time of the initial charge of sea water.

To use the equation simply plug in the value for the time and evaluate the formula.

Problem 3: The equation for the heat expenditure (q), in joules, was found to be,

$$q = \left(5 + 8 * e^{\left(-\frac{0.001*\pi}{F}\right)}\right) * 418600$$

Where (F) is the flow rate of the water, in m^3/min. To use this equation simply plug in the value of the time and compute the heat expenditure.

Below is a graph of the quantity of heat (q) and the flow rate (F). Also, the straight line (red) is the heat expenditure if the heat exchanger is not used. It has a value of 5441800 Joules compared to a maximum value of 4955000 Joules when the flow rate is at 0.02 m^3/min.